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# SOME THREE-DIMENSIONAL INCLUSION PROBLEMS IN ELASTICITY

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**Abstract**—The theory of potential functions is applied to solve a number of three-dimensional problems involving sheet-like inclusions embedded in elastic solids. Two types of inclusions are considered; namely, that of a rigid elliptical disk and a rigid sheet containing an elliptical hole. By varying the ellipticity of the disk and hole, certain information on the general character of the stresses around a plane inclusion of arbitrary shape may be obtained. More precisely, if reference is made to a suitable coordinate system, the functional forms of the stresses in the close neighborhood of the inclusion border can be expressed independently of uncertainties of both the inclusion geometry and of the applied stresses or displacements. In general, the intensification of the local stresses can be described by three parameters which may be used to establish criteria for the failure of the solid containing the inclusions.

## INTRODUCTION

DURING the past few decades, considerable attention has been devoted to the solution of two- and three-dimensional problems of stress concentrations around inclusions of a variety of shapes. Since the literature on this subject is extensive, only those works which are pertinent to the present study will be cited.

The problem of a thin rigid circular disk embedded in an infinite solid and subjected to a constant displacement normal to its plane was solved by Collins [1]. His results are equivalent to the slow steady motion of a rigid disk in a viscous fluid. In a recent paper, Keer [2] has considered a similar problem in which the disk is displaced in its own plane. The case of an infinite solid containing a rigid sheet with a circular hole was also discussed in [2]. The disturbance of an ellipsoidal inclusion in an otherwise uniform stress field was examined by Eshelby [3, 4]. In the limit as one of the principal axes of the ellipsoid vanishes, the solution to the problem of a flat elliptical disk may be deduced from the work in [3, 4]. The case when the ellipsoidal inclusion undergoes translational and rotational movements was considered by Lur'e [5].

For the purpose of assessing the strength degradation of solids due to the presence of disk-shaped inclusions, it is important to have a knowledge of the singular behavior of the stresses near the sharp edges of the inclusions. To this end, the present investigation is concerned primarily with the determination of stress solutions of the following boundary-value problems:

(1) A plane inclusion of elliptical shape in an otherwise uniform tensile field.
(2) Elliptical disk displaced in its own plane.

(3) Displacement given to a rigid sheet with an elliptical hole.
(4) Elliptically-shaped disk displaced out of its own plane.

Referring to a system of Cartesian coordinates $x$, $y$, $z$, the $z$-axis will be directed normal to the plane of discontinuity which is bounded by the ellipse

$$x^2/a^2+y^2/b^2 = 1, \qquad z = 0 \tag{1}$$

where $a$ and $b$ are the major and minor semi-axes of the ellipse, respectively. The center of the ellipse is located at the origin of the coordinate system. The rectangular components of displacement $u_x$, $u_y$, $u_z$ and stress $\sigma_{xx}$, $\sigma_{yy}$, ---, $\tau_{zx}$ are assumed to be continuously differentiable at all interior points of the solid and take definite values on either side of the ellipse except that on the periphery of the ellipse the stresses may become infinitely large. At large distances from the origin, all the stresses and displacements tend to zero for cases (2) to (4). The problem is to find a suitable solution of the Navier's equation of linear elasticity for a homogeneous, isotropic body.

In the absence of body forces, the displacement vector $\mathbf{u}$ is governed by the equation

$$\nabla^2\mathbf{u}+\frac{1}{1-2\nu}\nabla\nabla\,.\,\mathbf{u} = 0 \tag{2}$$

where $\nu$ is Poisson's ratio. The gradient and Laplacian operators in three-dimensions are denoted by $\nabla$ and $\nabla^2$, respectively. For problems exhibiting symmetry about the $xy$-plane, which contains the surface of discontinuity, the displacement vector $\mathbf{u}$ may be expressed in terms of a vector potential $\boldsymbol{\phi}$ with components $\phi_x$, $\phi_y$, $\phi_z$ and a scalar potential $\psi$ [6].

$$\mathbf{u} = \boldsymbol{\phi}+z\nabla\psi. \tag{3}$$

Hence, it is not difficult to verify that equation (2) can be satisfied by taking

$$\frac{\partial\psi}{\partial z} = -\frac{1}{3-4\nu}\nabla\,.\,\boldsymbol{\phi} \tag{4}$$

and

$$\nabla^2\phi = 0, \qquad \nabla^2\psi = 0$$

The displacement vectors for problems possessing symmetry with respect to the $yz$- and $zx$-planes may be obtained from equations (3) and (4) by cyclic permutation of the variables $x$, $y$, $z$. For instance, the representation

$$\mathbf{u} = \boldsymbol{\phi}'+x\nabla\psi', \qquad \frac{\partial\psi'}{\partial x} = -\frac{1}{3-4\nu}\nabla\,.\,\boldsymbol{\phi}' \tag{5}$$

applies to problems with symmetry about the $yz$-plane. In equation (5), $\boldsymbol{\phi}'$ and $\boldsymbol{\psi}'$ satisfy the Laplace equation in three-dimensions.

It should be mentioned that equation (3) or equation (5) is a special representation of the more general solution of Papkovitch [7]:

$$\mathbf{u} = 4(1-\nu)\mathbf{B}-\nabla(\mathbf{R}\,.\,\mathbf{B}+B_0) \tag{6}$$

where **R** is the position vector. Denoting the components of **B** by $B_x$, $B_y$, $B_z$, the Papkovitch functions are related to $\boldsymbol{\phi}$ and $\psi$ in equation (3) as

$$\phi_x = -\frac{\partial B_0}{\partial x}, \qquad \phi_y = -\frac{\partial B_0}{\partial y}, \qquad \phi_z = -\frac{\partial B_0}{\partial z}+(3-4\nu)B_z, \qquad \psi = B_z$$

and the two components $B_x$, $B_y$ are taken to be zero.

Once the displacements are known, the stress tensor $\boldsymbol{\sigma}$ follows directly from the stress–displacement relation

$$\boldsymbol{\sigma} = \mu\left[\frac{2\nu}{1-2\nu}(\nabla\cdot\mathbf{u})\mathbf{I}+\nabla\mathbf{u}+\mathbf{u}\nabla\right] \tag{7}$$

in which $\mu$ is the shear modulus of the material and **I** is the isotropic tensor.

## TRIAXIAL TENSION OF ELLIPTICAL DISK

Consider an infinite solid with an elliptical disk lying in the $xy$-plane. The $z$-axis pierces through the center of the disk whose surfaces are subjected to the displacements

$$\begin{aligned} Eu_x &= -[\sigma_1-\nu(\sigma_2+\sigma_3)]x, \\ Eu_y &= -[\sigma_2-\nu(\sigma_3+\sigma_1)]y,\ Eu_z = 0 \end{aligned} \tag{8}$$

for

$$z = 0 \quad \text{and} \quad x^2/a^2+y^2/b^2 \leq 1.$$

The Young's modulus is denoted by $E$. Now, the negative of the displacements in equation (8) correspond precisely to those of a uniform state of stress in a solid with the disk absent, i.e.,

$$\sigma_{xx} = \sigma_1, \qquad \sigma_{yy} = \sigma_2, \qquad \sigma_{zz} = \sigma_3, \qquad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0. \tag{9}$$

Superposition of the solutions of the two preceding problems will leave both faces of the disk free from displacement and will yield the result to the problem of a thin rigid elliptical disk in an otherwise uniform state of stress. Hence, it suffices to solve the non-trivial second fundamental problem owing to the boundary conditions given by equation (8).

Let the displacement components be expressed in terms of two harmonic functions which are similar in form to those used in formulating the analogous crack problem, i.e.,

$$\begin{aligned} u_x &= \frac{\partial F_1}{\partial x}-\frac{\partial F_2}{\partial x}-\frac{\partial \bar{F}_2}{\partial y} \\ u_y &= \frac{\partial F_1}{\partial y}+\frac{\partial F_2}{\partial y}+\frac{\partial \bar{F}_2}{\partial x} \\ u_z &= z\left(\frac{\partial^2 f_1}{\partial z^2}+\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial y^2}\right). \end{aligned} \tag{10}$$

The functions $F_j$ and $\bar{F}_2$ are given by

$$F_j = (3-4\nu)f_j + z\frac{\partial f_j}{\partial z}, \qquad (j = 1, 2)$$
$$\bar{F}_2 = \frac{2}{a^2-b^2}\left(a^2 y\frac{\partial f_2}{\partial x} - b^2 x\frac{\partial f_2}{\partial y}\right) \tag{11}$$

where

$$\nabla^2 f_j(x, y, z) = 0.$$

Those terms in equation (10) containing $f_1$ $(x, y, z)$ are derived from equation (3) by taking

$$\phi_x = (3-4\nu)\frac{\partial f_1}{\partial x}, \qquad \phi_y = (3-4\nu)\frac{\partial f_1}{\partial y}, \qquad \phi_z = 0, \qquad \psi = \frac{\partial f}{\partial z} \tag{12}$$

and those with $f_2$ $(x, y, z)$ is a particular solution of the problem of an ellipsoidal inclusion in an otherwise uniform field of stress as presented by Eshelby [3, 4]*.

Upon substituting of equation (10) into (7) gives the stress components

$$\frac{\sigma_{xx}}{2\mu} = -2\nu\left(\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}\right) + \frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 \bar{F}_2}{\partial x \partial y}$$
$$\frac{\sigma_{yy}}{2\mu} = -2\nu\left(\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}\right) + \frac{\partial F_1}{\partial y^2} + \frac{\partial F_2}{\partial y^2} + \frac{\partial^2 \bar{F}_2}{\partial x \partial y}$$
$$\frac{\sigma_{zz}}{2\mu} = (1-2\nu)\left(\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}\right)$$
$$\frac{\tau_{xy}}{\mu} = 2\frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 \bar{F}_2}{\partial x^2} - \frac{\partial^2 \bar{F}_2}{\partial y^2} \tag{13}$$
$$\frac{\tau_{xz}}{\mu} = 4(1-\nu)\frac{\partial^2 f_1}{\partial x \partial z} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 \bar{F}_2}{\partial y \partial z} + z\frac{\partial}{\partial x}\left(2\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}\right)$$
$$\frac{\tau_{yz}}{\mu} = 4(1-\nu)\frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 \bar{F}_2}{\partial x \partial z} + z\frac{\partial}{\partial y}\left(2\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}\right).$$

To determine the unknown functions $f_j$ $(x, y, z)$, $j = 1, 2$, the symmetrical form of ellipsoidal coordinates $\xi$, $\eta$, $\zeta$ will be employed. The rectangular coordinates $x$, $y$, $z$ of any point of the infinite space will be expressed in terms of the triply orthogonal system $\xi$, $\eta$, $\zeta$ in the form [8]

$$a^2(a^2-b^2)x^2 = (a^2+\xi)(a^2+\eta)(a^2+\zeta)$$
$$b^2(b^2-a^2)y^2 = (b^2+\xi)(b^2+\eta)(b^2+\zeta) \tag{14}$$
$$a^2b^2z^2 = \xi\eta\zeta$$

* More specifically, this particular solution follows from equation (3.1) in [4] by letting (in Eshelby's notation)

$$e_{11}^T = 1, \qquad e_{22}^T = -1, \qquad e_{33}^T = 0; \qquad e_{ij}^T = 0, \qquad i \neq j$$

where

$$\infty > \xi \geq 0 \geq \eta \geq -b^2 \geq \zeta \geq -a^2.$$

In the plane $z = 0^{\pm}$, the inside of the two-sided ellipse $x^2/a^2 + y^2/b^2 = 1$ is given by $\xi = 0$, and the outside by $\eta = 0$.

The boundary conditions, equation (8), may be satisfied if

$$f_j(x, y, z) = \frac{A_j}{2} \int_{\xi}^{\infty} \left[ \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1 \right] \frac{ds}{\sqrt{[Q(s)]}}, \qquad (j = 1, 2) \tag{15}$$

where

$$Q(s) = s(a^2+s)(b^2+s).$$

Apart from a multiplying constant, equation (15) represents the gravitational potential at an external point of a uniform elliptical disk [9]. Such functions have been used in obtaining the solution to some closely related crack problems [10].

For subsequent use, the following* partial derivatives are computed:

$$\begin{aligned} \frac{\partial f_j}{\partial x} &= \frac{2A_j}{a^3 k^2} [u - E(u)] x \\ \frac{\partial f_j}{\partial y} &= \frac{2A_j}{a^3 k^2 k'^2} \left[ E(u) - k'^2 u - k^2 \cdot \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} \right] y. \end{aligned} \tag{16}$$

The variable $u$ is related to the ellipsoidal coordinate $\xi$ by

$$\xi = a^2(\operatorname{sn}^{-2} u - 1)$$

and

$$E(u) = \int_0^u \operatorname{dn}^2 t \, dt.$$

The quantities sn$u$, cn$u$, ---, represent the Jacobian elliptic functions and $k$, $k'$ stand for

$$ak = (a^2 - b^2)^{\frac{1}{2}}, \quad ak' = b$$

Equations (8) render a system of two algebraic equations for the two unknown constants $A_1$ and $A_2$. This gives

$$\begin{aligned} A_1 &= \frac{a^3 k^2}{2E} \frac{\sigma_1[\nu M - N - (1-\nu)P] - \sigma_2[M - \nu N + (1-\nu)P] + \nu\sigma_3[M + N - 2P]}{2MN + P(M+N)} \\ A_2 &= \frac{a^3 k^2}{2E} \frac{\sigma_1(\nu M + N) - \sigma_2(M + \nu N) + \nu\sigma_3(M - N)}{2MN + P(M+N)} \end{aligned} \tag{17}$$

* The higher order derivatives of the function $f(x, y, z)$ can be found in a paper by Kassir and Sih [10].

in which $M$, $N$, and $P$ stand for

$$M = (3-4\nu)[K(k)-E(k)]$$
$$N = (3-4\nu)\left[\frac{E(k)}{k'^2}-K(k)\right] \tag{18}$$
$$P = \frac{2}{k^2}[(1+k'^2)K(k)-2E(k)].$$

Here, $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind associated with the modulus $k$, respectively.

Once the constants $A_1$ and $A_2$ are known, the displacements and stresses at any point of the solid may be computed. For instance, across the plane $z = 0$, it is found that

$$(\sigma_{zz})_{\xi=0} = \frac{4\mu(1-2\nu)}{ab^2k^2}\{-A_1k^2E(k)+A_2[2k'^2K(k)-(1+k'^2)E(k)\}$$
$$(\tau_{xz})_{\xi=0} = \frac{8\mu(1-\nu)(A_2-A_1)x}{a^3b}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}} \tag{19}$$
$$(\tau_{yz})_{\xi=0} = -\frac{8\mu(1-\nu)(A_1+A_2)y}{ab^3}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}.$$

Outside of the ellipse $x^2/a^2+y^2/b^2 = 1$, i.e., for $\eta = 0$. The shear stresses $\tau_{xz}$ and $\tau_{yz}$ vanish and

$$(\sigma_{zz})_{\eta=0} = \frac{4\mu(1-2\nu)}{ab^2}\left\{A_1\left[\frac{ab^2}{\sqrt{[Q(\xi)]}}-E(u)+\frac{\text{sn}u\,\text{cn}u}{\text{dn}\,u}\right]\right.$$
$$+\frac{A_2}{k^2}\left[2k'^2u-(1+k'^2)E(u)+k^2\frac{\text{sn}u\,\text{cn}u}{\text{dn}\,u}\right] \tag{20}$$
$$\left.-\frac{b^2A_2}{ak^2(\xi-\zeta)\sqrt{[Q(\xi)]}}[(a^2+b^2)(\xi+\zeta)+2(a^2b^2+\xi\zeta)]\right\}$$

Equations (19) and (20) show that the shear stresses $\tau_{xz}$ and $\tau_{yz}$ are unbounded on the boundary of the disk for $\xi = 0$ while the normal stress $\sigma_{zz}$ become singular on the edge of the disk for $\eta = 0$. The other components of the normal stress $\sigma_{xx}$ and $\sigma_{yy}$ are also singular on the edge of the disk for $\eta = 0$. Further, the stress exerted by the surrounding material on the disk in the $z$-direction vanishes when the material is incompressible.

In the limiting case of $a = b$, $E(k) = K(k) = \pi/2$, the constants $A_1$ and $A_2$ in equation (17) take the forms

$$A_1 = -\frac{a^3}{(3-4\nu)\pi E}[(1-\nu)(\sigma_1+\sigma_2)-2\nu\sigma_3]$$
$$A_2 = \frac{a^3}{\pi E}\frac{(1+\nu)(\sigma_1+\sigma_2)}{5-4\nu} \tag{21}$$

and equations (19) reduce to

$$\sigma_{zz} = \frac{(1-2\nu)[(1-\nu)(\sigma_1+\sigma_2)-2\nu\sigma_3]}{(3-4\nu)(1+\nu)}, \qquad z = 0^+, \qquad r < a$$

$$\sigma_{xz} = -8\mu(1-\nu)\frac{A_1-A_2}{a^3}\frac{(r/a)\cos\theta}{\sqrt{[1-(r^2/a^2)]}}, \qquad z = 0^+, \qquad r < a \tag{22}$$

$$\sigma_{yz} = -8\mu(1-\nu)\frac{A_1+A_2}{a^3}\frac{(r/a)\sin\theta}{\sqrt{[1-(r^2/a^2)]}}, \qquad z = 0^+, \qquad r < a$$

and $\sigma_{xz} = \sigma_{yz} = 0$ for $r > a$, $z = 0$. These values agree with those obtained by Collins [1]. For the special case $\sigma_1 = \sigma_2 = 0$, the shear stresses in equation (22) may be combined to yield

$$\sigma_{rz} = \mp\frac{8\nu(1-\nu)\sigma_3}{\pi(1+\nu)(3-4\nu)}\frac{r/a}{\sqrt{[1-(r^2/a^2)]}}; \qquad r < a, \qquad z = 0. \tag{23}$$

The plus and minus signs refer to the lower and upper faces of the disk, respectively.

## ELLIPTICAL DISK DISPLACED ALONG ITS MAJOR AXIS

Let an elliptical disk be embedded in an infinite solid and be placed in the $xy$-plane. The disk is displaced along its major axis by the amount $u_0$, a constant. The necessary boundary conditions are

$$\begin{aligned} &u_x = u_0; \qquad u_y = u_z = 0, \qquad \xi = 0 \\ &u_z = \tau_{xz} = \tau_{yz} = 0, \qquad \eta = 0. \end{aligned} \tag{24}$$

The symmetry conditions suggest the following selection of potential functions:

$$\phi'_x = -(3-4\nu)g+\frac{\partial h}{\partial x}, \qquad \phi'_y = \frac{\partial h}{\partial y}, \qquad \phi'_z = \frac{\partial h}{\partial z}, \qquad \psi' = g \tag{25}$$

where $\phi'_x$, $\phi'_y$, $\phi'_z$ are the rectangular components of the vector $\boldsymbol{\phi}$ in equation (5). The functions $g(x, y, z)$ and $h(x, y, z)$ satisfy the Laplace equations

$$\nabla^2 g(x, y, z) = 0, \qquad \nabla^2 h(x, y, z) = 0$$

Putting equation (25) into (5), it is found that

$$u_x = -4(1-\nu)g+\frac{\partial G}{\partial x}, \qquad u_y = \frac{\partial G}{\partial y}, \qquad u_z = \frac{\partial G}{\partial z} \tag{26}$$

in which

$$G = xg+h.$$

From equation (7), the components of stress are obtained:

$$\frac{\sigma_{xx}}{2\mu} = -2(2-\nu)\frac{\partial g}{\partial x}+\frac{\partial^2 G}{\partial x^2}, \qquad \frac{\sigma_{yy}}{2\mu} = -2\nu\frac{\partial g}{\partial x}+\frac{\partial^2 G}{\partial y^2},$$
$$\frac{\sigma_{zz}}{2\mu} = -2\nu\frac{\partial g}{\partial x}+\frac{\partial^2 G}{\partial z^2},$$
$$\frac{\tau_{xy}}{2\mu} = -2(1-\nu)\frac{\partial g}{\partial y}+\frac{\partial^2 G}{\partial x\partial y}, \qquad \frac{\tau_{yz}}{2\mu} = \frac{\partial^2 G}{\partial y\partial z}, \tag{27}$$
$$\frac{\tau_{zx}}{2\mu} = -2(1-\nu)\frac{\partial g}{\partial z}+\frac{\partial^2 G}{\partial x\partial z}.$$

The appropriate harmonic functions for this problem may be chosen as

$$g(x,y,z) = B_1\int_\xi^\infty \frac{\mathrm{d}s}{\sqrt{[Q(s)]}} = \frac{2B_1}{a}u,$$
$$h(x,y,z) = B_2 x\int_\xi^\infty \frac{\mathrm{d}s}{(a^2+s)\sqrt{[Q(s)]}} = \frac{2B_2}{a^3k^2}[u-E(u)]x. \tag{28}$$

Note that $h(x, y, z)$, except for the multiplying constant, represents the derivative of the gravitational potential at an external point of an elliptical disk with respect to $x$. For the purpose of evaluating the constants $B_1$ and $B_2$, the displacement component $u_z$ is computed:

$$u_z = -\frac{2x[\eta\zeta(a^2+\xi)(b^2+\xi)]^{\frac{1}{2}}}{ab(\xi-\eta)(\xi-\zeta)}\left[B_1+\frac{B_2}{a^2+\xi}\right].$$

The condition that $u_z$ vanishes everywhere on the plane $z = 0$ yields

$$B_2 = -a^2B_1. \tag{29}$$

By virtue of equations (24), (26) and (29) for $\xi = 0$, $B_1$ is found:

$$B_1 = -\frac{u_0}{2}\cdot\frac{ak^2}{[(3-4\nu)k^2+1]K(k)-E(k)}. \tag{30}$$

Knowing $B_1$ and $B_2$, the displacements and stresses at any point of the solid can be calculated. On the plane $z = 0$, the non-vanishing displacements are

$$(u_x)_{\eta=0} = -\frac{2B_1}{ak^2}\left\{[1+(3-4\nu)k^2]u-E(u)+\frac{a(kx)^2}{(\xi-\zeta)(a^2+\xi)}\sqrt{\left[\frac{\xi(b^2+\xi)}{a^2+\xi}\right]}\right\} \tag{31}$$
$$(u_y)_{\eta=0} = -\frac{2B_1xy}{\xi-\zeta}\cdot\sqrt{\left[\frac{\xi}{(a^2+\xi)(b^2+\xi)}\right]},$$

and the stresses are

$$(\tau_{xz})_{\xi=0} = \frac{8\mu(1-\nu)B_1}{ab}(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}}$$
$$(\sigma_{zz})_{\eta=0} = -\frac{4\mu(1-2\nu)B_1x}{\xi-\zeta}\cdot\sqrt{\left[\frac{b^2+\xi}{\xi(a^2+\xi)}\right]}. \tag{32}$$

Both $\tau_{xz}$ and $\sigma_{zz}$ are singular on the border of the ellipse $x^2/a^2+y^2/b^2 = 1$, while $\tau_{yz} = 0$ everywhere on the plane $z = 0$.

When $a = b$, $K = E = \pi/2$, equation (30) simplifies to the form

$$B_1 = -\frac{2au_0}{\pi(7-8\nu)}.$$

It can be verified that for $r > a$, $z = 0$, $\xi \to r^2-a^2$, and $u \to \sin^{-1}(a/r)$, equations (31) and (32) are in agreement with equations (23) and (24) in [2], respectively, except for*

$$(\sigma_{zz})_{z=0} = \frac{8\mu(1-2\nu)}{\pi(7-8\nu)}\left(\frac{u_0}{a}\right)\cdot\frac{\cos\theta}{(r/a)\sqrt{[(r/a)^2-1]}}, \qquad r > a \tag{33}$$

where $u_0$ corresponds to $\Delta$ in [2].

The foregoing method of solution may also be used to solve the problem of an elliptical disk displaced in an arbitrary direction by a constant amount, say $\delta_0$. If $\omega$ denotes the angle between the $x$-axis and the direction along which the disk is caused to move, then the boundary conditions, equation (24), may be generalized:

$$u_x = \delta_0\cos\omega, \qquad u_y = \delta_0\sin\omega, \qquad u_z = 0, \qquad \xi = 0$$

$$u_z = \tau_{xz} = \tau_{yz} = 0, \qquad \eta = 0.$$

The displacements are expressible in terms of four harmonic functions as

$$u_x = -4(1-\nu)g_1+\frac{\partial G_0}{\partial x}, \qquad u_y = -4(1-\nu)g_2+\frac{\partial G_0}{\partial y}, \qquad u_z = \frac{\partial G_0}{\partial z}$$

in which

$$G_0 = G_1+G_2, \qquad G_1 = xg_1+h_1, \qquad \text{and } G_2 = yg_2+h_2$$

To satisfy the Laplace equations in three dimensions, $g_j(x, y, z)$ and $h_j(x, y, z)$ are taken in the forms

$$g_j(x, y, z) = C_j\int_\xi^\infty \frac{ds}{\sqrt{[Q(s)]}}, \qquad j = 1, 2$$

$$h_1(x, y, z) = D_1x\int_\xi^\infty \frac{ds}{(a^2+s)\sqrt{[Q(s)]}}$$

$$h_2(x, y, z) = D_2y\int_\xi^\infty \frac{ds}{(b^2+s)\sqrt{[Q(s)]}}.$$

* Equation (33) may also be derived directly from equation (20) in [2] if the order of integration and differentiation is properly observed as follows:

$$(\sigma_{zz})_{z=0} = \frac{1}{2}(1-2\nu)\frac{\partial}{\partial x}\left\{\lim_{z\to 0}\int_{-a}^{+a}\frac{f(t)\,dt}{\sqrt{[r^2+(z+it)^2]}}\right\}, \qquad f(t) = -\frac{8\mu u_0}{\pi(7-8\nu)}.$$

Carrying out the integration gives

$$(\sigma_{zz})_{z=0} = -\frac{8\mu(1-2\nu)u_0}{\pi(7-8\nu)}\frac{\partial}{\partial x}\left[\sin^{-1}\left(\frac{a}{r}\right)\right]$$

$$= \frac{8\mu(1-2\nu)u_0}{\pi(7-8\nu)}\left(\frac{a}{r}\right)(r^2-a^2)^{-\frac{1}{2}}\cos\theta.$$

Hence, the factor $(1-\nu)$ in equation (24) of [2] should be replaced by $\cos\theta$.

Since the displacement $u_z$ vanishes for $z = 0$, the constants $D_j$ may be expressed in terms of $C_j$:

$$D_1 = -a^2 C_1, \qquad D_2 = -b^2 C_2.$$

The remaining unknowns, say $C_j$ ($j = 1, 2$), can be evaluated from the boundary conditions yet to be satisfied and the solution of the problem is essentially complete.

## DISPLACEMENT OF RIGID SHEET WITH ELLIPTICAL HOLE

Suppose that two semi-infinite solids are bonded perfectly to a thin rigid sheet with an elliptical opening through which the solids are connected. The sheet is allowed to move in the plane $z = 0$ by a constant amount parallel to the $x$-axis. The equivalent condition is to specify a constant shear stress $\tau_{zx} = \tau_0$ for $\xi = 0$. For this problem, the following conditions must be satisfied:

$$\begin{aligned} &u_x = u_y = 0, \qquad \eta = 0; \qquad u_z = 0, \qquad z = 0 \\ &\tau_{yz} = 0, \qquad \tau_{zx} = \tau_0, \qquad \xi = 0 \end{aligned} \tag{34}$$

The problem may be formulated in terms of a single function $p(x, y, z)$ which is related to $\boldsymbol{\phi}$ and $\psi$ in equations (3) and (4) as

$$\phi_x = -(3-4\nu)\frac{\partial p}{\partial z}, \qquad \phi_y = \phi_z = 0, \qquad \psi = \frac{\partial p}{\partial x}$$

where

$$\nabla^2 p(x, y, z) = 0.$$

The representation of the components of displacement as given by Trefftz [6] is

$$u_x = -(3-4\nu)\frac{\partial p}{\partial z} + z\frac{\partial^2 p}{\partial x^2}, \qquad u_y = z\frac{\partial^2 p}{\partial x \partial y}, \qquad u_z = z\frac{\partial^2 p}{\partial x \partial z}. \tag{35}$$

The stresses corresponding to equation (35) are given by

$$\begin{aligned} &\frac{\sigma_{xx}}{2\mu} = \frac{\partial}{\partial x}\left[-(3-2\nu)\frac{\partial p}{\partial z} + z\frac{\partial^2 p}{\partial x^2}\right], \frac{\sigma_{yy}}{2\mu} = \frac{\partial}{\partial x}\left[-2\nu\frac{\partial p}{\partial z} + z\frac{\partial^2 p}{\partial y^2}\right], \\ &\frac{\sigma_{zz}}{2\mu} = \frac{\partial}{\partial x}\left[(1-2\nu)\frac{\partial p}{\partial z} + z\frac{\partial^2 p}{\partial z^2}\right], \frac{\tau_{xy}}{\mu} = \frac{\partial}{\partial y}\left[-(3-4\nu)\frac{\partial p}{\partial z} + 2z\frac{\partial^2 p}{\partial x \partial y}\right], \\ &\frac{\tau_{yz}}{\mu} = \frac{\partial^2}{\partial x \partial y}\left[p + 2z\frac{\partial p}{\partial z}\right], \frac{\tau_{zx}}{\mu} = -(3-4\nu)\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2}{\partial x^2}\left[p + 2z\frac{\partial p}{\partial z}\right]. \end{aligned} \tag{36}$$

On the plane $z = 0$, equations (34) require that

$$\frac{\partial p}{\partial z} = 0, \qquad \eta = 0 \qquad \frac{\partial^2 p}{\partial x^2} - (3-4\nu)\frac{\partial^2 p}{\partial z^2} = \frac{\tau_0}{\mu}, \qquad \xi = 0. \tag{37}$$

The first condition in equations (37) is satisfied automatically by taking

$$p(x, y, z) = \frac{C}{2}\int_{\xi}^{\infty}\left[\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{s}-1\right]\frac{\mathrm{d}s}{\sqrt{[Q(s)]}}$$

while the second condition yields

$$2\mu C = \frac{a^3k^2k'^2\tau_0}{k'^2K(k)+[(3-4\nu)k^2-k'^2]E(k)}.$$

Once $p(x, y, z)$ is determined, the displacements and stresses throughout the solid can be computed from equations (35) and (36).

For $z = 0$, both $u_y$ and $u_z$ vanish and

$$(u_x)_{\xi=0} = -\frac{2C(3-4\nu)}{ab}(1-x^2/a^2-y^2/b^2)^{\frac{1}{2}}, \qquad (u_x)_{\eta=0} = 0.$$

The stresses on the plane $z = 0$ are

$$\begin{aligned}(\sigma_{zz})_{\xi=0} &= -\frac{4\mu(1-2\nu)Cx}{a^3b}(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}}\\ (\tau_{yz})_{\eta=0} &= -\frac{2\mu Cxy}{(\xi-\zeta)\sqrt{[Q(\xi)]}}\\ (\tau_{zx})_{\eta=0} &= 2\mu C\left\{-\frac{3-4\nu}{ab^2}\left[\frac{ab^2}{\sqrt{[Q(\xi)]}}-E(u)+\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}\right]\right.\\ &\quad\left.+\frac{u-E(u)}{a^3k^2}-\frac{x^2}{(\xi-\zeta)(a^2+\xi)}\sqrt{\left[\frac{b^2+\xi}{\xi(a^2+\xi)}\right]}\right\}\end{aligned} \qquad (38)$$

and

$$(\sigma_{zz})_{\eta=0} = (\tau_{yz})_{\xi=0} = 0, \qquad (\tau_{zx})_{\xi=0} = \tau_0.$$

Using L' Hospital's rule, the constant $C$ for a circular hole, $a = b$, may be recovered:

$$C = \frac{2a^3\tau_0}{\pi\mu(7-8\nu)}$$

Aside from a couple of misprints, $(u_x)_{\xi=0}$, $(\tau_{yz})_{\eta=0}$, and $(\tau_{zx})_{\eta=0}$ check with those given by equations (41) and (42) in [2] if $\tau_0$ is identified with $\sigma_0$. The expression for

$$(\sigma_{zz})_{z=0} = -\frac{8(1-2\nu)}{\pi(7-8\nu)}\frac{r/a}{\sqrt{[1-(r/a)^2]}}\tau_0\cos\theta$$

fails to agree with that of [2] for the same reason as mentioned earlier in the footnote on p. 233.

## AXIAL DISPLACEMENT OF ELLIPTICAL DISK

If a thin rigid disk of elliptical shape is given a constant displacement $w_0$ normal to its plane, then

$$u_x = u_y = 0, \qquad z = 0; \qquad u_z = w_0, \qquad \xi = 0 \qquad (39)$$

which suggests that

$$\phi_x = \phi_y = 0, \qquad \phi_z = -(3-4\nu)q, \qquad \psi = q. \tag{40}$$

Inserting equation (40) into (3), the result is

$$u_x = z\frac{\partial q}{\partial x}, \qquad u_y = z\frac{\partial q}{\partial y}, \qquad u_z = -(3-4\nu)q + z\frac{\partial q}{\partial z}. \tag{41}$$

From equation (7), it is further found that

$$\begin{aligned}
\frac{\sigma_{xx}}{2\mu} &= -2\nu\frac{\partial q}{\partial z} + z\frac{\partial^2 q}{\partial x^2}, \qquad \frac{\sigma_{yy}}{2\mu} = -2\nu\frac{\partial q}{\partial z} + z\frac{\partial^2 q}{\partial y^2},\\
\frac{\sigma_{zz}}{2\mu} &= -2(1-\nu)\frac{\partial q}{\partial z} + z\frac{\partial^2 q}{\partial z^2}, \qquad \frac{\tau_{xy}}{2\mu} = z\frac{\partial^2 q}{\partial x \partial y}\\
\frac{\tau_{yz}}{2\mu} &= -(1-2\nu)\frac{\partial q}{\partial y} + z\frac{\partial^2 q}{\partial y \partial z}, \ \frac{\tau_{zx}}{2\mu} = -(1-2\nu)\frac{\partial q}{\partial x} + z\frac{\partial^2 q}{\partial x \partial z}.
\end{aligned} \tag{42}$$

The only unknown function $q(x, y, z)$ satisfying

$$\nabla^2 q(x, y, z) = 0$$

can be taken in the form

$$q(x, y, z) = D\int_\xi^\infty \frac{ds}{\sqrt{[Q(s)]}} = \frac{2D}{a}u. \tag{43}$$

Equations (39), (41) and (43) may be combined to give

$$D = -\frac{aw_0}{2(3-4\nu)K(k)}.$$

Calculating for the derivatives of $q(x, y, z)$ with respect to $x, y, z$, i.e.,

$$\begin{aligned}
\frac{\partial q}{\partial x} &= \frac{aw_0 x}{(3-4\nu)(\xi-\eta)(\xi-\zeta)K(k)} \cdot \sqrt{\left[\frac{\xi(b^2+\xi)}{a^2+\xi}\right]},\\
\frac{\partial q}{\partial y} &= \frac{aw_0 y}{(3-4\nu)(\xi-\eta)(\xi-\zeta)K(k)} \cdot \sqrt{\left[\frac{\xi(a^2+\xi)}{b^2+\xi}\right]}\\
\frac{\partial q}{\partial z} &= \frac{w_0(\eta\zeta)^{\frac{1}{2}}}{(3-4\nu)b(\xi-\eta)(\xi-\zeta)K(k)} \cdot \sqrt{[(a^2+\xi)(b^2+\xi)]}
\end{aligned}$$

and so on ---, the non-trivial displacements and stresses for $z = 0$ are

$$(u_z)_{\xi=0} = w_0, \qquad (u_z)_{\eta=0} = \frac{w_0}{K(k)} \cdot [u]_{\eta=0}$$

and

$$(\sigma_{zz})_{z=0\pm} = \mp\frac{4\mu(1-\nu)w_0}{(3-4\nu)b\,K(k)}(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}}, \qquad \xi = 0$$

$$\begin{bmatrix}(\tau_{xz})_{z=0^+}\\ (\tau_{yz})_{z=0^+}\end{bmatrix} = -\frac{2\mu(1-2\nu)w_0}{(3-4\nu)\xi^{\frac{1}{2}}(\xi-\zeta)k\,K(k)}\begin{bmatrix}\sqrt{[(a^2+\zeta)(b^2+\xi)]}\\ \sqrt{\{(a^2+\xi)[-(b^2+\zeta)]\}}\end{bmatrix}, \qquad \eta = 0 \tag{44}$$

in which $-(b^2+\zeta)$ is a positive definite quantity. The notations $z = 0^+$ and $z = 0^-$ refer to the upper and lower faces of the disk, respectively.

The force exerted by the elastic solid to oppose the displacement of the elliptical disk may be found from the integral

$$F_z = \int_\Sigma \int [(\sigma_{zz})_{z=0^+} - (\sigma_{zz})_{z=0^-}] \, dx \, dy. \tag{45}$$

The region $\Sigma$ is bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. Substituting equation (44) into (45), $F_z$ is obtained:

$$\begin{aligned} F_z &= -\frac{8\mu(1-\nu)w_0}{(3-4\nu)b\,K(k)} \int_\Sigma \int (1 - x^2/a^2 - y^2/b^2)^{-\frac{1}{2}} \, dx \, dy \\ &= -\frac{16\pi\mu(1-\nu)aw_0}{(3-4\nu)K(k)}. \end{aligned} \tag{46}$$

In the limit as $a \to b$, equation (46) reduces to Collin's solution [1] for a circular disk.

## THREE-DIMENSIONAL STRESSES NEAR INCLUSION BORDER

For the purpose of establishing possible failure criteria, the stresses near the border of a plate-like inclusion will be investigated. It is convenient to introduce a rectangular cartesian coordinate system $n$, $t$, $z$ such that the origin of this system traverses the periphery of the inclusion. The $zn$-, $nt$-, and $tz$-planes are known, respectively, as the normal, rectifying and osculating planes to the curve which will be taken in the form of an ellipse.

In the immediate vicinity of the inclusion border, the ellipsoidal coordinates $\xi$, $\eta$, $\zeta$ can be expressed in terms of the polar coordinates $r$, $\theta$ defined in the $nz$-plane, where $r$ is the radial distance measured from the edge of the inclusion and $\theta$ is the angle between $r$ and the $n$-axis. The required relationships of $\xi$, $\eta$, $\zeta$ to $r$, $\theta$ are*

$$\begin{aligned} \xi &= \frac{2abr}{(a^2 \sin^2\phi + b^2 \cos^2\phi)^{\frac{1}{2}}} \cos^2 \frac{\theta}{2} \\ \eta &= -\frac{2abr}{(a^2 \sin^2\phi + b^2 \cos^2\phi)^{\frac{1}{2}}} \sin^2 \frac{\theta}{2} \\ \zeta &= -(a^2 \sin^2\phi + b^2 \cos^2\phi). \end{aligned} \tag{47}$$

In equation (47), $r$ is assumed to be small in comparison with $a$ (or $b$) and $\phi$ is the angle appearing in the parametric equations of the ellipse, i.e.,

$$x = a\cos\phi, \qquad y = b\sin\phi.$$

Since the derivation of the local stresses is similar to those given by Kassir and Sih [10] for the three-dimensional crack problem, the detail calculations will be omitted here. By means of equation (47) and the appropriate equations for finding the stresses, the following results

* A detailed derivation of equation (47) is given in [10].

are obtained:

$$
\begin{aligned}
\sigma_{nn} &= +\frac{k_1}{\sqrt{(2r)}}\cos\frac{\theta}{2}\left(3-2\nu-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
&\quad +\frac{k_2}{\sqrt{(2r)}}\sin\frac{\theta}{2}\left(2\nu+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)+O(1)\\
\sigma_{zz} &= -\frac{k_1}{\sqrt{(2r)}}\cos\frac{\theta}{2}\left(1-2\nu-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
&\quad +\frac{k_2}{\sqrt{(2r)}}\sin\frac{\theta}{2}\left(2-2\nu-\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)+O(1)\\
\sigma_{tt} &= +\frac{k_1}{\sqrt{(2r)}}\,.\,2\nu\cos\frac{\theta}{2}+\frac{k_2}{\sqrt{(2r)}}2\nu\sin\frac{\theta}{2}+O(1) \qquad (48)\\
\tau_{nt} &= -\frac{k_3}{\sqrt{(2r)}}\cos\frac{\theta}{2}+O(1)\\
\tau_{nz} &= +\frac{k_1}{\sqrt{(2r)}}\sin\frac{\theta}{2}\left(2-2\nu+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\
&\quad +\frac{k_2}{\sqrt{(2r)}}\cos\frac{\theta}{2}\left(1-2\nu+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+O(1)\\
\tau_{tz} &= +\frac{k_3}{\sqrt{(2r)}}\sin\frac{\theta}{2}+O(1).
\end{aligned}
$$

Although these stresses were derived from the solution of an elliptically-shaped inclusion, they are in general valid for a plane inclusion of arbitrary shape. Moreover, the inclusion-border stress fields for the four preceding boundary-value problems are included in equation (48) as special cases.

Now, it is significant to observe that equation (48) is composed of the linear sum of three distinct stress fields each of which can be associated with a different mode of deformation. Referring to Figs. 1(a) through 1(c), the intensity of the local stresses at the point $P$ caused by the movements of the inclusion in the $n$-, $z$-, and $t$- directions are governed, respectively, by the three parameters $k_1$, $k_2$ and $k_3$. These three modes of displacements are necessary and sufficient to describe all the possible displacements of the inclusion. It will be shown subsequently that the parameters $k_j$ $(j = 1, 2, 3)$ depend only upon the prescribed stresses or displacements and the inclusion geometry. The singular behavior of the inclusion-border stresses is the same as that for a sharp crack. In other words, the $1/\sqrt{r}$ type of stress singularity is preserved. However, unlike the crack problem, the angular distribution of the stresses is a function of the Poisson's ratio of the elastic solid.

A close examination of the stress expressions in equation (48) reveals that $\sigma_{nn}$, $\sigma_{zz}$, and $\tau_{nz}$ correspond precisely to those obtained by Sih [11]* for a rigid line inclusion under the

* The stresses $\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ given by equation (48) in [11] should be transformed into rectangular components $\sigma_{xx}$, $\sigma_{yy}$, $\tau_{xy}$ in accordance with

$$\sigma_{xx}+\sigma_{yy} = \sigma_{rr}+\sigma_{\theta\theta}$$

$$\sigma_{yy}-\sigma_{xx}+2i\tau_{xy} = e^{-2i\theta}(\sigma_{\theta\theta}-\sigma_{rr}+2i\tau_{r\theta})$$

For $\kappa = 3-4\nu$, the functional forms of $\sigma_{xx}$, $\sigma_{yy}$, $\tau_{xy}$ correspond to $\sigma_{nn}$, $\sigma_{zz}$, $\tau_{nz}$ in this paper, respectively.

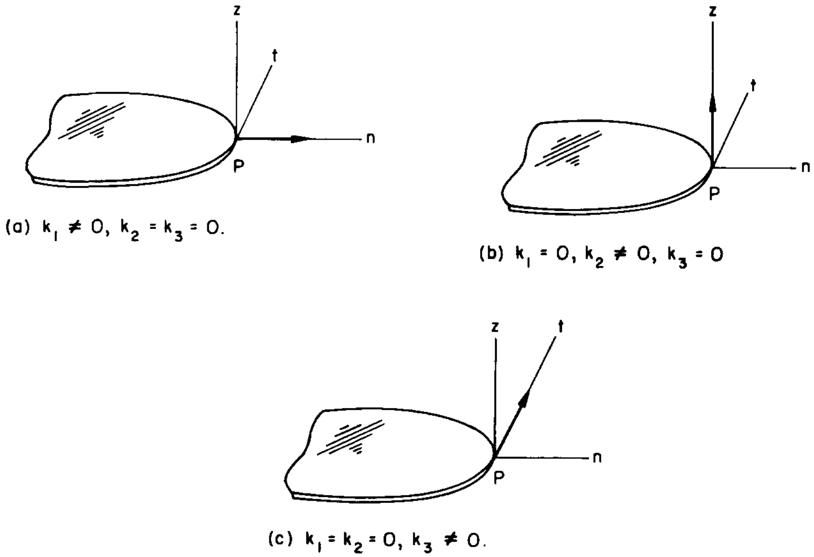


FIG. 1. The basic modes of plane inclusion displacements.

conditions of plane strain. In fact, the stress component $\sigma_{tt}$ is equal to $v(\sigma_{nn}+\sigma_{zz})$, a condition which is well known in the analysis of plane strain problems. The shear stresses $\tau_{nt}$ and $\tau_{tz}$ can be identified with the two-dimensional problem of a line inclusion subjected to longitudinal or out-of-plane shear loads. Hence, the stress state around a plane inclusion in three-dimensions is locally one of plane strain combined with longitudinal shear.

In general, the three parameters $k_j$ $(j = 1, 2, 3)$ will occur simultaneously over the inclusion border. They may be interpreted as a measure of the elevation of stresses due to the presence of thin rigid inclusions embedded in elastic solids. From equation (48), the formulas

$$
\begin{aligned}
k_1 &= \frac{1}{1-2v}\lim_{r\to 0}\sqrt{(2r)}(\sigma_{zz})_{\theta=0} \\
k_2 &= \frac{1}{1-2v}\lim_{r\to 0}\sqrt{(2r)}(\tau_{nz})_{\theta=0} \\
k_3 &= \lim_{r\to 0}\sqrt{(2r)}(\tau_{tz})_{\theta=0}
\end{aligned}
\tag{49}
$$

are obtained. Equation (49) may be applied to evaluate $k_j$ for the boundary-value problems solved earlier. Following the work of Kassir and Sih [10], it is found that

1. *Triaxial tension*

$$
\begin{aligned}
k_1 &= +\frac{4\mu(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{4}}}{(ab)^{\frac{3}{2}}}\left[\frac{b^2\cos^2\phi-a^2\sin^2\phi}{b^2\cos^2\phi+a^2\sin^2\phi}A_2-A_1\right] \\
k_2 &= 0, k_3 = -\frac{16\mu(1-v)A_2\sin\phi\cos\phi}{(ab)^{\frac{3}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{4}}}
\end{aligned}
\tag{50}
$$

in which $A_1$ and $A_2$ are given by equation (17).

2. *Parallel displacement*

$$k_1 = -\frac{2\mu ak^2u_0}{[(3-4\nu)k^2+1]K(k)-E(k)}\left(\frac{b}{a}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{4}}\cos\phi, \qquad k_2 = 0$$

$$k_3 = \frac{4\mu(1-\nu)ak^2u_0}{[(3-4\nu)k^2+1]K(k)-E(k)}\left(\frac{a}{b}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{4}}\sin\phi \tag{51}$$

3. *Rigid sheet*

$$k_1 = +\frac{2bk^2\tau_0}{[(3-4\nu)k^2-k'^2]E(k)+k'^2K(k)}\left(\frac{b}{a}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{4}}\cos\phi, \qquad k_2 = 0$$

$$k_3 = \frac{(3-4\nu)ak^2\tau_0}{[(3-4\nu)k^2-k'^2]E(k)+k'^2K(k)}\left(\frac{b}{a}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{4}}\sin\phi \tag{52}$$

4. *Axial displacement*

$$k_1 = 0, \qquad k_2 = -\frac{2\mu w_0}{(3-4\nu)K(k)}\left(\frac{a}{b}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{4}}, \qquad k_3 = 0 \tag{53}$$

It is interesting to note that $k_j$ are not constants but functions of position. Equation (53) is associated with the local displacement shown in Fig. 1(b). The displacement modes pertaining to the results in equations (50) through (52) are more complicated since for $0 < \phi < \pi/2$ the inclusion border experiences a combination of the movements illustrated in Figs. 1(a) and 1(c). In particular, the parameters $k_1$ and $k_3$ in equations (51) and (52) attain their maximum values at $\phi = 0$ and $\phi = \pi/2$, respectively.

For problems involving all three parameters $k_j$ $(j = 1, 2, 3)$, it is possible to postulate a criterion of failure for rigid inclusions in the form

$$f_{cr} = f(k_1, k_2, k_3)$$

which states that failure of the material surrounding the inclusion occurs when the combination of $k_1$, $k_2$, and $k_3$ attains some critical value.

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**Абстракт**—Теория потенциальных функций приложена к расчету некоторого числа трехмерных задач, касающихся полос в виде инклюзий находящихся в упругих телах. Рассматриваются два типа инклюзий, а именно: жесткий элиптический диск и жесткая полоса, содержаюая элиптическое отверствие. Изменяя элиптичность диска и отверствия можна получить некоторые сведения об общем характере касающихся напряжений вокруг плоской инклюзии произвольной формы. Волее точно, в случае подбора надлежащей координатной системы, можна представить функциональные формы напряжений в близкой окрестности границы инклюзии, независимо от ее геометрии и приложенных напряжений или перемещений. Вообще, интенсификация локальных напряжений может быть описана трема параметрами, которые используются для сосдания критерев разрушения твердого тела, обладающего инклюзиами.